The 6th Olympiad of Metropolises

Mathematics

Solutions. Day 2

Problem 4. Six real numbers $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are given. For each triplet of distinct numbers of those six Vitya calculated their sum. It turned out that the 20 sums are pairwise distinct; denote those sums by

$$s_1 < s_2 < s_3 < \cdots < s_{19} < s_{20} .$$

It is known that $x_2 + x_3 + x_4 = s_{11}$, $x_2 + x_3 + x_6 = s_{15}$ and $x_1 + x_2 + x_6 = s_m$. Find all possible values of m.

Answer: $m = 7$.

Solution. Observation 1. Any sum containing x_1 is less than any sum not containing x_1.

Indeed, note that $x_2 + x_3 + x_4$ is the smallest among all 10 sums not containing x_1. Since it is equal to s_{11}, all the other 9 sums that do not contain x_1 must be equal to $s_{12}, s_{13}, \ldots, s_{20}$ in some order. Hence, all smaller sums, from s_1 to s_{10}, must contain x_1.

Observation 2. Consider the sums without x_1. Among them any sum containing x_6 is greater than any sum not containing x_6.

Indeed, similarly, note that $x_2 + x_3 + x_6$ is the smallest among all 6 sums that do not contain x_1, but contain x_6. Since it is equal to s_{15}, all the other 5 sums containing x_6, but not x_1, must be equal to $s_{16}, s_{17}, \ldots, s_{20}$ in some order. Then all smaller sums, from s_{11} to s_{14}, do not contain x_6.

Observation 3. Any pair of triplets with the sum $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ is distributed symmetrically, i.e., if the sum of a triplet is s_i, the sum of the complementary triplet would be s_{21-i}.

(This one is obvious.)

Now we are ready to finish the solution. Consider $x_3 + x_4 + x_5$.

- According to observation 1 this sum is larger than all 10 sums containing x_1.
- By observation 2 this sum is less than any of 6 sums without x_1 but with x_6.
- Among the other 4 sums (without both x_1 and x_6) this sum is clearly the greatest.

Thus we have $x_3 + x_4 + x_5 = s_{14}$. Then $x_1 + x_2 + x_6 = s_7$ by observation 3.

Remark 1. It is easy to construct such numbers. For example, the numbers $-32, 1, 2, 4, 8, 16$ are appropriate. Their sums in all combinations are different, and claims 1 and 2 hold true; these claims are equivalent to the conditions $x_2 + x_3 + x_4 = s_{11}$ and $x_2 + x_3 + x_6 = s_{15}$.

Remark 2. The condition of the problem allows one to order all the sums, except for two pairs. Denoting $X_{ijk} = x_i + x_j + x_k$, one can prove that

$$X_{123} < X_{124} < (X_{125}, X_{134}) < X_{135} < X_{145} < X_{126} < X_{136} < X_{146} < X_{156} <$$
$$X_{234} < X_{235} < X_{245} < X_{345} < X_{236} < X_{246} < (X_{346}, X_{256}) < X_{356} < X_{456}\,. \quad \square$$

Problem 5. There is a safe that can be opened by entering a *secret code* consisting of n digits, each of them is 0 or 1. Initially, n zeros were entered, and the safe is closed (so, all zeros is not the secret code).

In one attempt, you can enter an arbitrary sequence of n digits, each of them is 0 or 1. If the entered sequence matches the secret code, the safe will open. If the entered sequence matches the secret code in more positions than the previously entered sequence, you will hear a click. In any other cases the safe will remain locked and there will be no click.

Find the smallest number of attempts that is sufficient to open the safe in all cases.

Answer: n.

Solution. Example. We present an algorithm opening the safe in n attempts.

By $A = (a_1, a_2, \dots, a_n)$ denote the secret code. Also set

$$A_k = (a_1, a_2, \dots, a_{k-1}, 1, 0, 0, \dots, 0)\,,$$

in particular, $A_0 = (0, 0, 0, 0, \dots, 0)$.

Let us show by induction on k for $k = 1, 2, \dots, n$ that on the k-th attempt (if the safe was not opened earlier) we can enter the sequence A_k (for this purpose, it is sufficient to find a_k on the k-th attempt).

This gives the required algorithm: indeed, if m is the maximal index for which $a_m = 1$, then $A = A_m$, and the safe is opened on the m-th attempt.

Base $k = 1$ is trivial.

Induction step. Let us prove the step $k \to k+1$. Assume that the safe is still closed after the k-th attempt.

Note that A_k differs of A_{k-1} in the k-th position, and possibly in one more, the $(k-1)$-th position, but in this position A_k matches A. Hence, if $a_k = 0$, then A_k is not “closer” (by the number of positions, in which the sequences match) to A than A_{k-1}. If $a_k = 1$, then A_k is “closer” to A than A_{k-1}. Thus we hear the click after the k-th attempt if and only if $a_k = 1$. Therefore, after the k-th attempt we know a_k.

Estimate. After each attempt, we will estimate the number of *possible* (i.e., not contradicting the outcomes of all past attempts, but also not tried yet) variants for the secret code. Initially, there are exactly $2^n - 1$ possible variants (all sequences of length n consisting of zeros and ones, except for the zero sequence). Assume that after the k-th attempt there are at least $2^{n-k} - 1$ possible variants for the code (for $k = 0$ it is already shown). Then we will prove that after the $(k+1)$-th attempt, regardless of the entered sequence, at least one of the outcomes will leave at least $2^{n-k-1} - 1$ possible variants.

Let b be the sequence that was entered on the $(k+1)$-th attempt. Let us divide the possible (before this attempt) variants into 3 groups: coinciding with B (either one or zero such variants); those for which a click should be heard; and those for which there will be no click. (Each variant will fall into exactly one such group.) Let us take from the last two groups the one with the largest number of variants, and then with the corresponding outcome we will have at least $\frac{1}{2}\big((2^{n-k} - 1) - 1\big) = 2^{n-k-1} - 1$ possible variants.

Thus, after $n - 1$ attempts, there is an outcome with at least one remaining possible variant, which means that the safe will still be closed. □

Another way to prove estimate. Let us prove that $n - 1$ attempts will not be sufficient.

Assume the contrary: let there be a written algorithm of opening the safe in no more than $n - 1$ attempts (steps). This algorithm should have the following structure. At the 1st step, some sequence B is entered. If the safe has opened, then we declare "success" and finish. Otherwise, at the 2nd step, the algorithm has 2 branches: in case the click was heard, B_1 is entered, otherwise B_2 is. At each next step, in case of failure, we have two variants of the entered sequence.

In general, no more than 2^{k-1} sequences are entered in all branches of the algorithm at the kth step. In total, no more than $1 + 2 + 4 + \cdots + 2^{n-2} = 2^{n-1} - 1$ different sequences are entered in all branches of the algorithm. But since each of the $2^n - 1$ sequences (except the zeros) may turn out to be the secret code, there will be a case in which such an algorithm will not handle the safe. □

Problem 6. Let $ABCD$ be a tetrahedron and suppose that M is a point inside it such that $\angle MAD = \angle MBC$ and $\angle MDB = \angle MCA$. Prove that

$$MA \cdot MB + MC \cdot MD < \max(AD \cdot BC, AC \cdot BD)\,.$$

Solution. Lemma.

$$\angle AMD + \angle BMC + \angle AMC + \angle BMD > 2\pi\,.$$

Proof of the lemma. Indeed, let the line DM intersect the plane ABC at point E, and the line BE intersect the segment AC at point F (fig. 1). Then we have $\angle AMC + \angle BMC = \angle AMF + (\angle CMF + \angle BMC) > \angle AMF + \angle BMF = (\angle AMF + \angle FME) + \angle BME > \angle AME + \angle BME = (\pi - \angle AMD) + (\pi - \angle BMD)$, hence $\angle AMC + \angle BMC + \angle AMD + \angle BMD > 2\pi$. *Lemma is proven.*

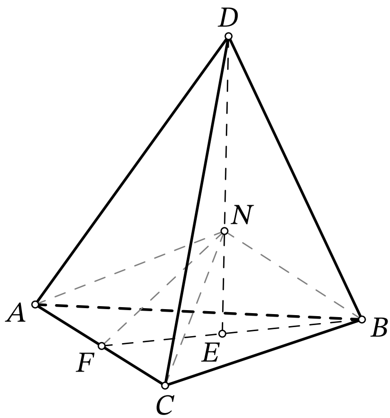

Figure 1: for the solution of problem 6

According to lemma, $(\angle AMD + \angle BMC) + (\angle AMC + \angle BMD) > 2\pi$, which means either $\angle AMC + \angle BMD > \pi$ or $\angle AMD + \angle BMC > \pi$. Without loss of generality, we will assume that the first case holds.

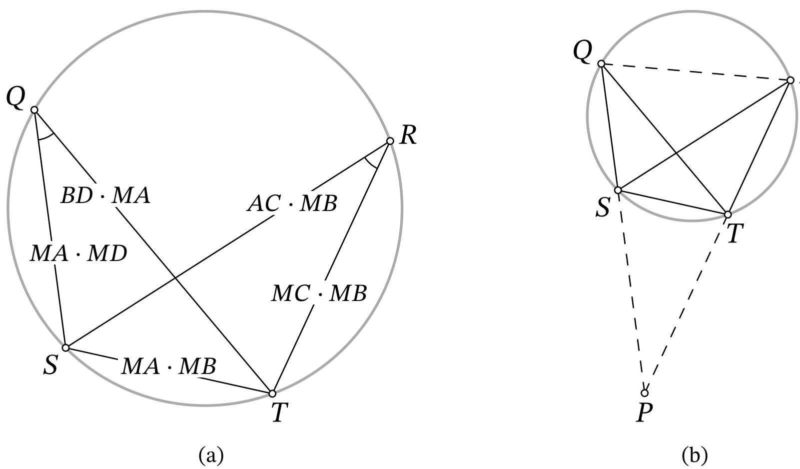

Figure 2: for the solution of problem 6

We multiply the lengths of the sides of the triangles AMC and BMD by BM and AM, respectively, and from the resulting triangles we compose the shape shown in fig. 2a. Since $\angle SQT = \angle MDB = \angle MCA = \angle TRS$, the points S, Q, R and T lie on one circle.

We have $\angle RTS + \angle QST = \angle AMC + \angle BMD > \pi$, therefore, the rays QS and RT intersect at some point P, as in fig. 2b. According to Ptolemy's theorem, we have

$$QR \cdot MA \cdot MB + MA \cdot MD \cdot MC \cdot MB = AC \cdot MB \cdot BD \cdot MA\,,$$

therefore,

$$QR + MC \cdot MD = AC \cdot BD\,. \tag{1}$$

The similarity of triangles PQR and PTS implies

$$\frac{QR}{ST} = \sqrt{\frac{S(PQR)}{S(PST)}} > 1\,. \tag{2}$$

From (1) and (2) we obtain $AC \cdot BD > ST + MC \cdot MD = MA \cdot MB + MC \cdot MD$. □

Another solution. After we reduce the problem to the case $\angle AMC + \angle BMD > \pi$, we can finish the solution differently.

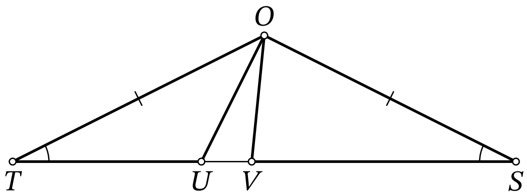

Figure 3: for another solution of problem 6

Let us construct triangle OUV with $\angle OUV = \pi - \angle AMC$ and $\angle OVU = \pi - \angle BMD$. There exist points T, S on the line UV such that $\triangle TUO \sim \triangle CMA$ and $\triangle SVO \sim \triangle DMB$; the location of points will be as shown on fig. 3.

From the equality of the angles $\angle UTO = \angle VSO$ we obtain that the triangle TOS is isosceles. Also

$$OT^2 = OU^2 + TU \cdot US \quad \text{and} \quad OT^2 = OV^2 + TV \cdot VS$$

(for example, since $-TU \cdot US$ is the power of point U with respect to the circle centered in O with radius $OT = OS$; and similarly for V).

Without loss of generality, we assume $OV \leq OU$; then

$$OT \cdot OS = OT^2 = OU^2 + TU \cdot US > OU \cdot OV + TU \cdot VS\,,$$

which by similarities is equivalent to $AC \cdot BD > MA \cdot MB + MC \cdot MD$. □