The 5th Olympiad of Metropolises

Mathematics

Solutions. Day 1

Problem 1. In a triangle ABC with a right angle at C, the angle bisector AL (where L is on segment BC) intersects the altitude CH at point K. The bisector of angle BCH intersects segment AB at point M. Prove that $CK = ML$. *(Alexey Doledenok)*

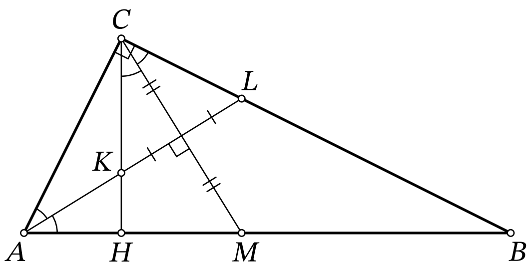

Figure 1: for the solution of problem 1

Solution. Since $\angle BCH = 90^\circ - \angle ABC = \angle BAC$, then

$$\angle ACM = 90^\circ - \angle BCM = 90^\circ - \angle BAC/2 = 90^\circ - \angle CAL\,,$$

therefore the bisectors AL and CM are perpendicular (fig. 1). In the triangle ACM the line AL contains the bisector and the altitude, so AL is the perpendicular bisector to the segment CM. Similarly, in the triangle CKL the line CM contains the angle bisector and the altitude, so CM is the perpendicular bisector to KL. Then $CKML$ is a rhombus, so $CK = ML$. □

Problem 2. Does there exist a positive integer n such that all its digits (in the decimal system) are greater than 5, while all the digits of n^2 are less than 5? *(Nazar Agakhanov)*

Answer: no.

Solution. Assume that, on the contrary, there exists such n. Suppose that n consists of k digits, thus

$$10^k > n \geq \underbrace{666...6}_{k} = \frac{2}{3}(10^k - 1)\,.$$

If $n = \underbrace{666...6}_{k}$, then the last digit of n^2 equals 6. Otherwise,

$$10^k > n > \frac{2}{3} \cdot 10^k\,,$$

and, hence,

$$10^{2k} - 1 = \underbrace{999...9}_{2k} \geq n^2 > \frac{4}{9} \cdot 10^{2k} > \frac{4}{9} \cdot \underbrace{999...9}_{2k} = \underbrace{444...4}_{2k} .$$

Finally,

$$\underbrace{999...9}_{2k} \geq n^2 > \underbrace{444...4}_{2k},$$

which means that n^2 consist of $2k$ digits, and all its digits can not be less or equal than 4. □

Problem 3. Let $n > 1$ be a given integer. The Mint issues coins of n different values a_1, a_2, ..., a_n, where each a_i is a positive integer (the number of coins of each value is unlimited). A set of values $\{a_1, a_2, \dots, a_n\}$ is called *lucky,* if the sum $a_1 + a_2 + \cdots + a_n$ can be collected in a unique way (namely, by taking one coin of each value).

(a) Prove that there exists a lucky set of values $\{a_1, a_2, \dots, a_n\}$ with

$$a_1 + a_2 + \cdots + a_n < n2^n.$$

(b) Prove that every lucky set of values $\{a_1, a_2, \dots, a_n\}$ satisfies

$$a_1 + a_2 + \cdots + a_n > n2^{n-1}.$$

(Ilya Bogdanov)

Solution of the part (a). We will show that the values $a_i = 2^n - 2^{n-i}$, $i = 1, 2, \dots, n$, make a lucky set. Notice here that

$$S = \sum_{i=1}^{n} a_i = n2^n - \sum_{j=0}^{n-1} 2^j = (n-1)2^n + 1 < n2^n .$$

Assume now that S is collected by some coins,

$$S = \sum_{k=1}^{p} a_{i_k} = p2^n - \sum_{k=1}^{p} 2^{j_k} ,$$

where $j_k = n - i_k \in \{0, 1, 2, \dots, n-1\}$. Since $S > (n-1)2^n$, we get $p \geq n$, and

$$\sum_{k=1}^{p} 2^{j_k} = (p - n + 1)2^n - 1 \equiv -1 \pmod{2^n}.$$

Without loss of generality, assume that $j_1 \leq j_2 \leq \cdots \leq j_p$. We claim that $j_k \leq k - 1$ for all $k = 1, 2, \dots, n$. Arguing indirectly, choose a minimal $k \leq n$ such that $j_k \geq k$. Then

$$-1 \equiv \sum_{s=1}^{k-1} 2^{j_s} \pmod{2^k},$$

which is impossible since

$$0 \le \sum_{s=1}^{k-1} 2^{j_s} \le \sum_{s=1}^{k-1} 2^{s-1} = 2^{k-1} - 1 .$$

This contradiction verifies the claim.

Finally we get

$$(p-n+1)2^n - 1 = \sum_{k=1}^{n} 2^{j_k} + \sum_{k=n+1}^{p} 2^{j_k} \le \sum_{k=1}^{n} 2^{k-1} + \sum_{k=n+1}^{p} 2^{n-1} = 2^n - 1 + (p-n)2^{n-1} ,$$

or $(p-n)2^n \le (p-n)2^{n-1}$. This may happen only when $p = n$, and all inequalities turn into equalities, which yields $j_k = k-1$. In other words, $S = a_1 + \cdots + a_n$ is indeed the unique way to collect S by the suggested coins. □

Remark 1. A different way to verify the claim is to take the multiset $\{2^{j_1}, 2^{j_2}, \dots, 2^{j_p}\}$ and modify it repeatedly by merging two copies of some 2^j into a single instance of 2^{j+1}. When the process stops, all numbers in the multiset are distinct, and the sum is still congruent to -1 modulo 2^k, so that the multiset should contain all powers 2^0, ..., 2^{k-1}. Thus, at the end of the process, the multiset contains k numbers smaller than 2^k, and therefore it contained at least k such numbers before the process.

Remark 2. There are several working examples. E.g., one may set $a_1 = 2^{n-1}$ and $a_i = 2^n + 2^{i-2}$ for $i = 2, 3, \dots, n$.

Solution of the part (b). Let us show that $a_1 \ge 2^{n-1}$ in any lucky collection of n coins $a_1 < \cdots < a_n$; this immediately yields $S = a_1 + \cdots + a_n > na_1 \ge n2^{n-1}$.

Denote $a = a_1$, and let Σ be the multiset of all sums which can be collected using coins a_2, a_3, ..., a_n, each taken at most once; thus, Σ consists of exactly 2^{n-1} numbers (some of which may be equal to each other), the minimal of those numbers is 0, while the maximal one is $a_2 + a_3 + \cdots + a_n$.

Assume that Σ contains two numbers, $S_1 \ge S_2$, which are congruent modulo a, so that $S_1 = S_2 + at$ for some integer $t \ge 0$. Then there exists an alternative way to collect S by the coins, namely

$$S = (S - S_1) + S_2 + at$$

(which means that we take the coins a_1, ..., a_n, remove the collection with sum S_1, add the collection with sum S_2, and add t coins of value a). This violates the luckiness.

Thus, Σ contains 2^{n-1} numbers pairwise incongruent modulo a, which yields $a \ge 2^{n-1}$. □