

# The 3rd Olympiad of Metropolises

## Day 2. Solutions

**Problem 4.** Let  $1 = d_0 < d_1 < \dots < d_m = 4k$  be all positive divisors of  $4k$ , where  $k$  is a positive integer. Prove that there exists  $i \in \{1, \dots, m\}$  such that  $d_i - d_{i-1} = 2$ .

(Ivan Mitrofanov)

*Solution 1.* Assume the contrary. This means that if  $d$  and  $d+2$  both divide  $4k$ , then  $d+1$  also divides  $4k$ . Note that if  $a$  divides  $4k$  and  $a$  is not divisible by 4, then  $2a$  also divides  $4k$ . Using the properties above, we start from the pair  $(1, 2)$  and find more pairs  $(a, a+1)$  such that both  $a$  and  $a+1$  divide  $4k$  and both  $a$  and  $a+1$  are not divisible by 4.

Let  $(a, a+1)$  be a pair of divisors of  $4k$  such that 4 divides neither  $a$  nor  $a+1$ . Then  $2a$  and  $2a+2$  divide  $4k$ , hence  $2a+1$  divides  $4k$ . One of  $2a$  or  $2a+2$  is not divisible by 4, hence in one of pairs  $(2a, 2a+1)$ ,  $(2a+1, 2a+2)$  both numbers divide  $4k$ , but are not divisible by 4.

Apply this procedure to the new pair, and so on. Thus, starting from the pair  $(1, 2)$  we obtain pairs  $(2, 3)$ ,  $(5, 6)$ ,  $(10, 11)$ , etc. At each step the sum of numbers in pair increases, hence we obtain an infinite set of divisors of  $4k$ . A contradiction.  $\square$

*Solution 2.* Assume the contrary. Let  $t$  be the minimal positive integer that does not divide  $4k$ . Then  $1, 2, \dots, t-1$  divide  $4k$ , while  $t$  and  $t+1$  do not (otherwise  $t-1$  and  $t+1$  would be two consecutive divisors of  $4k$ ).

It follows that  $t$  and  $t+1$  are prime powers, otherwise one of them would be a product of two coprime multiples less than  $t$  and would therefore divide  $4k$ . One of them is a power of 2 that we denote as  $2^m$ ,  $m \geq 3$ . The other has a form of  $2^m + \varepsilon$ ,  $\varepsilon = \pm 1$ .

Observe that  $2^{m-1}$  divides  $4k$ , since  $2^m$  is the minimal even non-divisor of  $4k$ , and that  $3 \cdot 2^{m-2} + \varepsilon$  divides  $4k$ , since it is odd and less than  $2^m + \varepsilon$ , which is the minimal odd non-divisor of  $4k$ . Also note that 3 divides  $4k$ .

It follows that  $3 \cdot 2^{m-1}$  and  $2 \cdot (3 \cdot 2^{m-2} + \varepsilon)$  also divide  $4k$ . Hence, the number  $3 \cdot 2^{m-1} + \varepsilon$  between them divides  $4k$  too. But now  $4k$  is divided by  $2 \cdot (3 \cdot 2^{m-1} + \varepsilon)$ , just as  $4 \cdot (3 \cdot 2^{m-2} + \varepsilon)$ . The number  $3 \cdot (2^m + \varepsilon)$  is between them, and must be a divisor of  $4k$  as well. We conclude that  $2^m + \varepsilon$  divides  $4k$ , contradiction.  $\square$

**Problem 5.** Ann and Max play a game on a  $100 \times 100$  board.

First, Ann writes an integer from 1 to 10 000 in each square of the board so that each number is used exactly once.

Then Max chooses a square in the leftmost column and places a token on this square. He makes a number of moves in order to reach the rightmost column. In each move the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Max pays Ann the number of coins equal to the number written in that square.

Max wants to pay as little as possible, whereas Ann wants to write the numbers in such a way to maximise the amount she will receive. How much money will Max pay Ann if both players follow their best strategies? *(Lev Shabanov)*

*Answer:* 500 000 coins.

*Solution. Lower bound / Ann's strategy.* First we will prove that Ann can get at least 500 000 coins. Suppose Ann has arranged the numbers in the way depicted on the fig. 1.

1	200	201	400	...	9800	9801	10000
2	199	202	399	...	9799	9802	9999
3	198	203	398	...	9798	9803	9998
4	197	204	397	...	9797	9804	9997
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
98	103	298	303	...	9703	9898	9903
99	102	299	302	...	9702	9899	9902
100	101	300	301	...	9701	9900	9901

Figure 1: for the solution of problem 5.

Consider a path constructed by Max. For every integer  $1 \leq n \leq 50$  there are two squares of the path in the columns  $2n - 1$  and  $2n$ , respectively, which are adjacent by a move of the token. It is easy to see that the sum of the numbers in such squares is at least  $200(2n - 1)$ . We obtain that the cost of this path is at least  $200(1 + 3 + 5 + \dots + 99) = 500 000$  coins.

*Upper bound / Max's strategy.* Now consider an arbitrary arrangement of the numbers. Then exclude the square with the greatest number in each column. The least of the numbers in excluded squares is not less than 100, the second least is not less than 200, etc, the greatest number in excluded square is 10 000. Hence, the sum of the excluded numbers is at least  $100 + 200 + \dots + 10 000 = 505 000$ , while the sum of

numbers in the whole square is 50 005 000. It means that the sum of the remaining numbers is at most 49 500 000.

The 9900 squares which are left can be split into 99 distinct paths. The first path consists of the lowest squares in each column which are not excluded, the second path consists of the lowest squares in each column which are not excluded and not included into the first path, etc. The last path will include the uppermost squares in each column which are not excluded. It is easy to see that all 99 paths are proper paths that Max's token can follow, because any two squares in two adjacent columns either lie in the same row or in two adjacent rows.

The total cost of the constructed 99 paths is not greater than 49 500 000, therefore one of these paths costs at most 500 000 coins. Thus in every arrangement of numbers Max can pay less than or equal to 500 000 coins.

We proved that Max can pay 500 000 coins or less while Ann can make Max pay at least 500 000 coins, so the answer is 500 000.  $\square$

*Alternate proof of the upper bound.* Split our board into 50 horizontal rectangles  $2 \times 100$ . Since the sum of numbers in the whole board is 50 005 000, it is possible to choose a rectangle with the sum of at most 1 000 100. Take a square with the minimum number in each column of the chosen rectangle. These squares form a path from the square in the left column to the square in the right column; denote the cost of this path as  $S$ . In each column of the rectangle, the minimal number is less than the other number by at least 1. Hence, the total sum of the numbers in the rectangle is at least  $2S + 100$ . It follows that  $2S + 100 \leq 1 000 100$  and  $S \leq 500 000$ .  $\square$

**Problem 6.** The incircle of a triangle  $ABC$  touches the sides  $BC$  and  $AC$  at points  $D$  and  $E$ , respectively. Suppose  $P$  is the point on the shorter arc  $DE$  of the incircle such that  $\angle APE = \angle DPB$ . The segments  $AP$  and  $BP$  meet the segment  $DE$  at points  $K$  and  $L$ , respectively. Prove that  $2KL = DE$ . *(Dušan Djukić)*

*A property of the symmedian.* The symmedian of a triangle from one of its vertices is defined as the reflection of the median from that vertex about the bisector from the same vertex (fig. 2). In the following solutions we will use a well-known property of the symmedian, namely that it passes through the intersection of the tangents to the circumcircle of the triangle taken at the other two vertices.

*Solution 1.* Denote by  $F$  the tangency point of the incircle with the side  $AB$ , and by  $M$  and  $N$  respectively the midpoints of the segments  $EF$  and  $DF$  (fig. 3). Since  $PB$  is the symmedian in the triangle  $DPF$ , we have  $\angle KPE = \angle DPB = \angle NPF$ . Moreover,  $\angle PEK = \angle PED = \angle PFN$ , so  $\triangle PEK \sim \triangle PFN$ . Analogously,  $\triangle PDL \sim \triangle PFM$ . Now we obtain  $EK = FN \cdot \frac{PE}{PF} = \frac{DF \cdot PE}{2PF}$  and similarly  $DL = FM \cdot \frac{PD}{PF} = \frac{EF \cdot PD}{2PF}$ , so  $EK + DL = \frac{DF \cdot PE + EF \cdot PD}{2PF} = \frac{1}{2}DE$  by Ptolemy's theorem. Therefore,  $KL = DE - EK - DL = \frac{1}{2}DE$ .  $\square$

*Solution 2.* Denote by  $F$  the tangency point of the incircle with the side  $AB$ . Con-

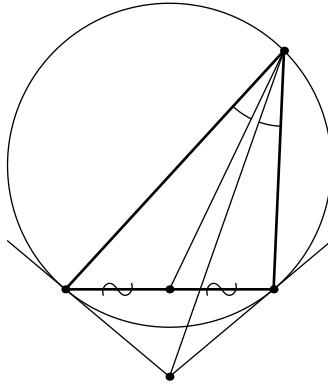


Figure 2: for the solution of problem 6.

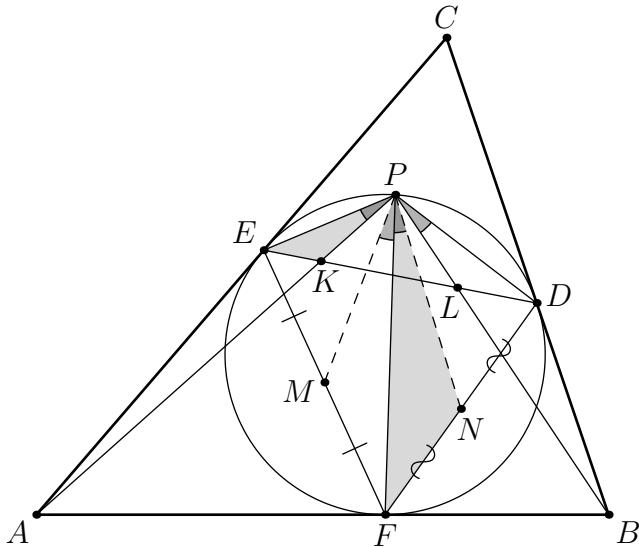


Figure 3: for the solution of problem 6.

sider a point  $S$  on the segment  $ED$  such that  $\angle PSE = \angle PDF$  and  $\angle PSD = \angle PEF$  (fig. 4). Triangles  $PSE$  and  $PDF$  are similar because  $\angle PED = \angle PFD$ . Since  $\angle DPB = \angle KPE$  and  $PB$  is a symmedian in triangle  $PDF$ , the line  $PK$  must be a median in the triangle  $PSE$ . It follows that  $EK = KS$ . Similarly, we have  $DL = LS$ . Therefore,  $2KL = 2KS + 2SL = EK + KS + SL + LD = ED$ .  $\square$

*Solution 3.* Again, denote by  $F$  the tangency point of the incircle with the side  $AB$ . Let the segments  $AP$  and  $BP$  intersect the incircle again at  $X$  and  $Y$ , respectively.

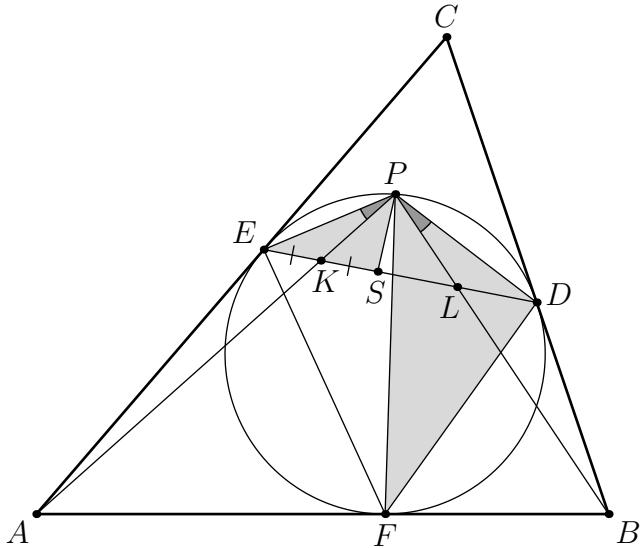


Figure 4: for the solution of problem 6.

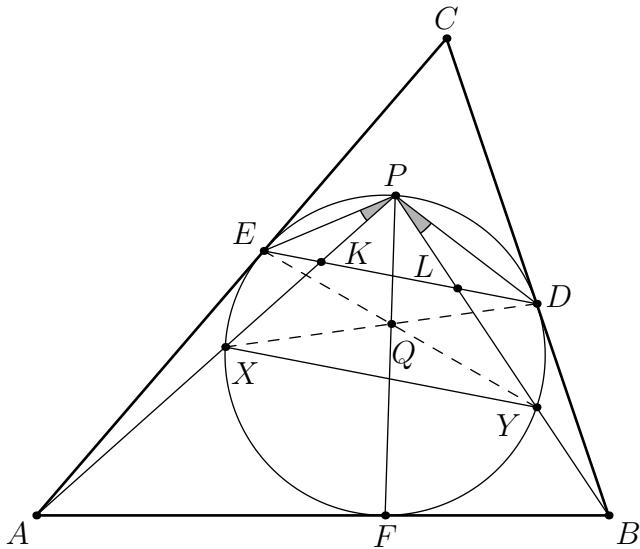


Figure 5: for the solution of the problem 6.

Since the arcs  $EX$  and  $DY$  are congruent, we have  $XY \parallel DE$ . Let the lines  $PF$  and  $EY$  meet at point  $Q$  (fig. 5). The quadrilaterals  $EPFX$  and  $YFPD$  are harmonic, and the projection with the center  $Q$  from the incircle to itself maps points  $E, P, F$

to  $Y, F, P$ , respectively. Therefore, this projection also sends  $X$  to  $D$ , i.e. the line  $XD$  also passes through  $Q$ .

Now let  $\ell$  be the line passing through the intersection point of  $XE$  and  $YD$ , parallel to  $DE$ . Consider the projective transformation preserving the incircle that takes line  $\ell$  to the infinity line. The lines  $DE$ ,  $XY$  and  $\ell$  will remain parallel under this transformation and the ratio  $KL/DE$  will not change. Moreover, the quadrilaterals  $XEPF$  and  $DYFP$  will remain harmonic. Thus we obtain fig. 6.

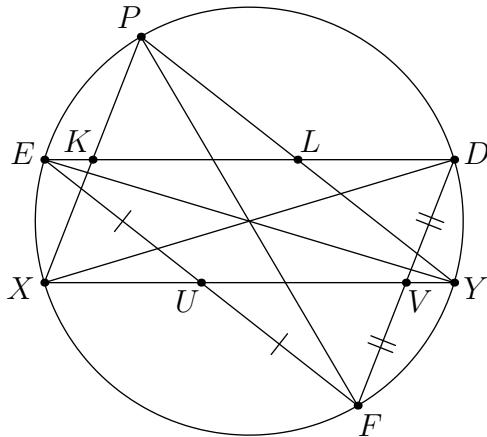


Figure 6: for the solution of the problem 6.

The quadrilateral  $XYDE$  is a rectangle and  $Q$  is the center, so  $P$  and  $F$  are diametrically opposite. Suppose the line  $XY$  meets the segments  $EF$  and  $DF$  at points  $U$  and  $V$ , respectively. Since  $XP$  is a symmedian in  $\triangle EXF$  and the shorter arcs  $EP$  and  $FY$  are equal, the line  $XU$  is a median in  $\triangle EXF$ , i.e.  $EU = UF$ . Analogously,  $DV = VF$ . It follows that the triangles  $EFD$  and  $UVF$  are proportional, so  $DE/2 = UV = KL$  by symmetry.  $\square$