2nd Olympiad of Metropolises

Day 1

Problem 1. Let $ABCD$ be a parallelogram in which the angle at B is obtuse and $AD > AB$. Points K and L are chosen on the diagonal AC such that $\angle ABK = \angle ADL$ (the points A, K, L, C are all different, with K between A and L). The line BK intersects the circumcircle ω of triangle ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$. *(Boyan Obukhov)*

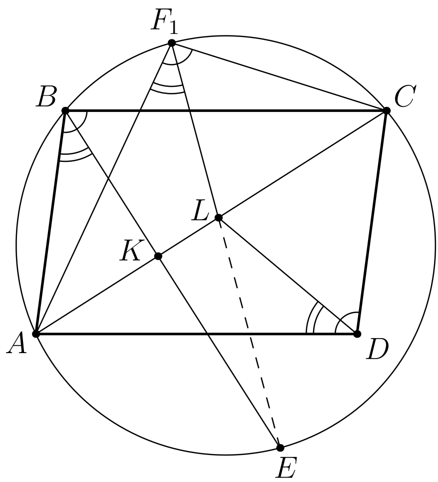

Figure 1: solution of problem 1.

Let F_1 be the point symmetric to D about line AC. (fig. 1).

Observe that $F_1 \in \omega$ since $\angle ABC = \angle ADC$ (the parallelogram property) and $\angle ADC = \angle AF_1C$ due to symmetry (thus $\angle ABC = \angle AF_1C$ and quadrilateral ABF_1C is inscribed in ω).

Since $ABCD$ is a parallelogram, we have $AB = CD$; also $CD = CF_1$ due to symmetry. Hence $AB = CF_1$ and ABF_1C is an equilateral trapezoid with $BF_1 \parallel AC$.

Next, due to symmetry we have $\angle AF_1L = \angle ADL$; also $\angle ADL = \angle ABK$ by the condition. Angles $\angle ABK$ and $\angle AF_1E$ are equal since they intercept the same arc. It follows that $\angle AF_1L = \angle AF_1E$, which means that points F_1, L, and E are collinear, so F and F_1 coincide. But we already have $BF_1 \parallel AC$, which concludes the solution. $\square$

Problem 2. In a country there are two-way non-stop flights between some pairs of cities. Any city can be reached from any other by a sequence of at most 100 flights.

Moreover, any city can be reached from any other by a sequence of an even number of flights. What is the smallest positive integer d for which one can always claim that any city can be reached from any other one by a sequence of an even number of flights not exceeding d?

(It is allowed to visit some cities or take some flights more than once.)

(Ilya Bogdanov)

Answer: $d = 200$.

Example. Join 201 cities in a cycle and consider a pair of neighboring cities in this cycle. Obviously, the shortest even path connecting these two cities contains 200 flights.

Estimate. Let A and B be two arbitrary cities. We prove that there exists an even path between A and B consisting of at most 200 flights. Consider the smallest even number $2k$ of flights from city A to city B and assume that $k > 100$. Let C be the k-th visited city after A. Note that one may reach C from A by $m \leq 100$ flights. If m and k have the same parity, this will allow shortening the even path from A to B, which is a contradiction. Thus m and k have different parities. Analogously, there is a route from C to B containing $n \leq 100$ flights, with n, k having different parity. These two routes allow reaching B from A via C by an even number $m + n \leq 200$ flights, a contradiction.

Estimate, another way. Let A and B be two arbitrary cities. For a city X, denote by $f(X)$ the minimal number of flights needed to reach B from X. Obviously $|f(X) - f(Y)| \leq 1$ for any flight XY. If $f(X) \neq f(Y)$ for every such flight, we see that flights are operated only between cities at which f takes different parities. It would imply that any route between two directly connected cities consists of an odd number of flights, a contradiction. Thus there exists a flight XY such that $f(X) = f(Y) =: b \leq 100$. Take a route from A to X containing $a \leq 100$ flights. We may reach B from A by $a + b$ flights, and also by $a + b + 1$ flights. One of these routes is even and contains at most 200 flights. □

Problem 3. Let $Q(t)$ be a quadratic polynomial with two distinct real zeros. Prove that there exists a non-constant polynomial $P(x)$ with the leading coefficient 1 such that the absolute values of all coefficients of the polynomial $Q(P(x))$ other than the leading one are less than 0.001. *(Dušan Djukić)*

Let $Q(t) = c\left((t-a)^2 - D\right)$, where a, c and D are real constants with $c \neq 0$ and $D > 0$. It suffices to construct a polynomial $R(x)$ with the leading coefficient equal to 1 such that all coefficients of the polynomial $R(x)^2 - D$ except for the leading one are less than $\delta = \frac{0.001}{|c|}$ in absolute value, and take $P(x) = R(x) + a$.

We are looking for the polynomial R in the form

$$R(x) = x^n + \varepsilon \cdot \sum_{i=1}^{m} \left(x^{n-a_i} + x^{a_i}\right) + b\,,$$

where m and $a_1 < a_2 < \cdots < a_m < n$ are positive integers, b and ε are real numbers. The monomials that occur in the expansion of $R(x)^2$, other than x^{2n}, x^n and the constant term, are x^{2n-a_i}, x^{2n-2a_i}, $x^{2n-a_i-a_j}$, x^{n+a_i}, $x^{n-a_i+a_j}$, x^{n-a_i}, $x^{a_i+a_j}$, x^{2a_i} and x^{a_i}. Let us set the numbers a_i in such a way that each of these monomials occurs only once (or twice, as in $x^{a_i+a_j} = x^{a_j+a_i}$). If $n > 3a_m$, this is equivalent to choosing numbers a_i so that the numbers a_i and $a_j + a_k$ are pairwise different for all i, j, k with $j \le k$. We may take $a_i = 3^{i-1}$ for $i = 1, \ldots, m$ and $n = 3^m + 1$.

Note that the coefficient of the polynomial $R(x)^2 - D$ at x^n is $2b + 2m\varepsilon^2$. Thus each coefficient of $R(x)^2 - D$ except for the leading one equals one of the numbers

$$0\,,\ 2\varepsilon\,,\ \varepsilon^2\,,\ 2\varepsilon^2\,,\ 2b\varepsilon\,,\ b^2 - D\,,\ 2b + 2m\varepsilon^2\,.$$

Taking $b = -\sqrt{D}$, $\varepsilon < \min\left\{\frac{\delta}{2b}, \frac{\delta}{2}, \sqrt{\frac{\delta}{2}}\right\}$ and $m = \lfloor \frac{-b}{\varepsilon^2} \rfloor$, we achieve that all these coefficients are less than δ in absolute value. $\square$