The 1st International Olympiad of Metropolises

September 2016

Solutions of day 2

Problem 4. A convex quadrilateral $ABCD$ has right angles at A and C. A point E lies on the extension of the side AD beyond D so that $\angle ABE = \angle ADC$. The point K is symmetric to the point C with respect to point A. Prove that $\angle ADB = \angle AKE$.
(Boyan Obukhov and Fedor Petrov)

The quadrilateral $ABCD$ is inscribed in the circle with diameter BD. Thus $\angle ADB = \angle ACB$ since both angles are subtended by the same arc. So, we have to prove that the angles BCA and AKE are equal, which in turn is equivalent to the claim that the lines BC and KE are parallel.

Note that $\angle BCD + \angle CDA = \angle BAD + \angle ABE < 180°$. It implies that the rays CB and DA have a common point which we denote by F. We have $\angle BFA = 90° - \angle ADC = 90° - \angle ABE = \angle BEA$. So BA is the altitude of the isosceles triangle FBE, this yields $FA = AE$. On the other hand $CA = AK$. So, the diagonals of the quadrilateral $FCEK$ have a common midpoint, i. e., $FCEK$ is a parallelogram. Therefore the lines FC and KE are indeed parallel as desired. □

Problem 5. Let $r(x)$ be a polynomial of odd degree with real coefficients. Prove that there exist only finitely many pairs of polynomials $p(x)$ and $q(x)$ with real coefficients satisfying the equation $(p(x))^3 + q(x^2) = r(x)$. *(Fedor Petrov)*

By replacing x by $-x$ and taking difference, we get $(p(x))^3 - (p(-x))^3 = r(x) - r(-x) = u(x)$, the polynomial $u(x)$ is non-zero, odd, and has the same degree as $r(x)$. We see that $p(x) - p(-x)$ is an odd divisor of $u(x)$. There are only finitely many divisors of $u(x)$ up to a constant factor. So, it suffices to check that for any fixed odd divisor $xa_0(x^2)$ of $u(x)$ there are only finitely many $p(x)$ such that $p(x) - p(-x)$ is proportional to $xa_0(x^2)$, i. e., $p(x)$ is of the form $\lambda xa_0(x^2) + b(x^2)$, where $\lambda \neq 0$ is some unknown constant and $b(t)$ is some unknown polynomial. For proving finiteness we may fix also the sign of λ. We have

$$u(x) = (p(x))^3 - (p(-x))^3 = 2xa_0(x^2) \cdot \left(3\lambda b^2(x^2) + \lambda^3 x^2 a_0^2(x^2)\right).$$

So, the polynomial $3\lambda b^2(t) + \lambda^3 t a_0^2(t)$ (we denoted $t = x^2$) is fixed: $3\lambda b^2(t) + \lambda^3 t a_0^2(t) = 3\lambda_0 b_0^2(t) + \lambda_0^3 t a_0^2(t)$ for some fixed solution $(\lambda_0, b_0(t))$. Rewrite it as

$$\lambda b^2 - \lambda_0 b_0^2 = \frac{\lambda_0^3 - \lambda^3}{3} t a_0^2(t)\,.$$

Dividing by λ_0 and factorizing the LHS as a difference of squares (which is possible in real numbers since λ and λ_0 have the same sign) we see that the pair of polynomials $\sqrt{\lambda/\lambda_0}b(t) \pm b_0(t)$ have the form $f(t)$, $\frac{\lambda_0^3-\lambda^3}{3\lambda_0}g(t)$ with $f(t)\cdot g(t) = t a_0^2(t)$. Again we may consider the case when $f(t)$ and $g(t)$ are fixed up to a constant factor: $f(t) = \tau f_0(t)$, $g(t) = \tau^{-1} g_0(t)$. We get

$$2b_0(t) = f(t) - \frac{\lambda_0^3 - \lambda^3}{3\lambda_0} g(t) = \tau f_0(t) - \tau^{-1}\frac{\lambda_0^3 - \lambda^3}{3\lambda_0} g_0(t)\,.$$

If this happens for two different pairs of values (τ, λ) and (τ', λ'), we may take the difference:

$$0 = (\tau - \tau')f_0(t) - \left(\tau^{-1}\frac{\lambda_0^3 - \lambda^3}{3\lambda_0} - (\tau')^{-1}\frac{\lambda_0^3 - (\lambda')^3}{3\lambda_0}\right) g_0(t)\,. \qquad (1)$$

If $\tau \neq \tau'$, it follows that $f_0(t)$ and $g_0(t)$ are proportional; but this is impossible, since their product $f_0(t)\cdot g_0(t) = t a_0^2(t)$ has odd degree. Otherwise, the coefficient of $f(t)$ in (1) is zero, hence coefficient of $g(t)$ is also zero, from which we obtain $(\lambda')^3 = \lambda^3$. It means that τ and λ are fixed, hence $f(t)$ and $g(t)$ are fixed, and there is at most one solution. Since on each step we diverged into finite number of cases, there is no more than a finite number of solutions totally. □

Problem 6. In a country with n cities, some pairs of cities are connected by one-way flights operated by one of two companies A and B. Two cities can be connected by more than one flight in either direction. An AB-word w is called *implementable* if there is a sequence of connected flights whose companies' names form the word w. Given that every AB-word of length 2^n is implementable, prove that every finite AB-word is implementable. (An AB-word of length k is an arbitrary sequence of k letters A or B; e. g. $AABA$ is a word of length 4.) *(Ivan Mitrofanov)*

Assume the contrary. Then there exist non-implementable words. Let $w = a_1a_2\ldots a_N$ be the shortest (or one of the shortest) non-implementable word. It is clear that $N > 2^n$. For any integer $0 \le i \le N$ denote by A_i the set of all cities that are the possible terminals of sequences of flights, that correspond to the word $a_1a_2\ldots a_i$. The set A_0 consists of all cities, A_N is empty. Since there are 2^n different subsets of the set of all cities, it follows by the pigeonhole principle that $A_i = A_j$ for some $i < j$.

Consider the word $w' = a_1a_2\ldots a_{i-1}a_ia_{j+1}a_{j+2}\ldots a_N$. Since it is shorter than w, we have that it is implementable. Let S be a sequence of flights implementing w'. By S_1 denote the sequence of the first i flights of S, by S_2 denote the sequence of the last $N - j$ flights of S, by T denote the endpoint of S_1. By construction, $T \in A_i$. Then, since $A_i = A_j$, it follows that there exists a sequence of flights S_3 implementing $a_1 a_2 \dots a_j$ and T is its terminal city.

But then the sequence of flights $S_3 S_2$ corresponds to $w = a_1 a_2 \dots a_N$ and w is implementable. This contradiction proves the statement of the problem. $\square$