The 1st International Olympiad of Metropolises

September 2016

Solutions of day 1

Problem 1. Find all positive integers n such that there exist n consecutive positive integers whose sum is a perfect square. *(Pavel Kozhevnikov)*

Answer: $n = 2^s m$, where m is any odd integer, and s is either 0 or odd.

Let $S(n,t) = (t+1) + (t+2) + \ldots + (t+n) = (2t+n+1)n/2$.

For odd n one may put $t = (n-1)/2$ and obtain $S(n,t) = n^2$.

Let n be even, $n = 2^s m$, where s is a positive integer, and m is odd. It follows that $2t+n+1$ is odd. Hence 2^{s-1} divides $S(n,t)$, while 2^s does not. This means that for even s the answer is negative. For odd s one may put $t = (mx^2 - n - 1)/2$ for some odd $x > n$ and obtain $S(n,t) = 2^{s-1}m^2x^2$. □

Problem 2. Let $a_1, \ldots, a_n$ be positive integers satisfying the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i} \le \frac{1}{2}.$$

Every year, the government of Optimistica publishes its *Annual Report* with n economic indicators. For each $i = 1, \ldots, n$, the possible values of the i-th indicator are $1, 2, \ldots, a_i$. The Annual Report is said to be *optimistic* if at least $n-1$ indicators have higher values than in the previous report. Prove that the government can publish optimistic Annual Reports in an infinitely long sequence.

(Ivan Mitrofanov, Fedor Petrov)

First we replace each a_i by a power of 2. For every $1 \le i \le n$, let k_i be the positive integer that satisfies $2^{k_i} \le a_i < 2^{k_i+1}$. Notice that $\sum\limits_{i=1}^{n} \frac{1}{2^{k_i}} < \frac{2}{a_i} \le 1$.

For every $1 \le i \le n$, we will choose a residue class A_i modulo 2^{k_i} in such a way that the classes $A_1, \ldots, A_n$ are pairwise disjoint. Without loss of generality we can assume that $k_1 \le k_2 \le \ldots \le k_n$. We choose $A_1, A_2, \ldots, A_n$ in this order. The residue class A_1 can be chosen arbitrarily. Suppose that we have already chosen the classes $A_1, \ldots, A_{i-1}$. In order to find the next class A_i, we require a residue modulo 2^{k_i} which is not used in any of $A_1, \dots, A_{i-1}$. Notice that for each $j < i$, the set A_j is the union of $2^{k_i - k_j}$ different residue classes modulo 2^{k_i}. As $\sum_{j=1}^{i-1} 2^{k_i - k_j} < 2^{k_i} \sum_{j=1}^{n} 2^{-k_j} < 2^{k_i}$, there are unused residues modulo 2^{k_i} which makes it possible to choose the new class A_i.

Now let us turn to the solution of the problem. For every $1 \le i \le n$, we will use only the first 2^{k_i} values of the i-th indicator. In the beginning let all indicators be equal to 1. In the y-th year, let the i-th indicator drop to 1 if $y \in A_i$, otherwise let the indicator increase by 1. Notice that the i-th indicator increases at most $2^{k_i} - 1$ times in a row, then drops to 1, so it never exceeds the bound $2^{k_i} \le a_i$ and therefore the values of the indicator form a valid report in every year. Since the residue classes $A_1, \dots, A_n$ are pairwise disjoint, at most one indicator drops in the same year, the reports keep optimistic. □

Problem 3. Let $A_1 A_2 \dots A_n$ be a cyclic convex polygon whose circumcenter is strictly in its interior. Let $B_1, B_2, \dots, B_n$ be arbitrary points on the sides $A_1 A_2$, $A_2 A_3, \dots, A_n A_1$, respectively, other than the vertices. Prove that

$$\frac{B_1 B_2}{A_1 A_3} + \frac{B_2 B_3}{A_2 A_4} + \dots + \frac{B_n B_1}{A_n A_2} > 1\,.$$

(Nairi Sedrakyan, David Harutyunyan)

Lemma 1. Suppose that a triangle without obtuse angle is inscribed in a circle of radius R. Then the perimeter of the triangle is greater than $4R$.
Proof. Let ABC be our triangle.
Assume that triangle ABC is right. Without loss of generality $\angle B = 90°$ and $AC = 2R$. Then $AB + BC + AC > AC + AC = 4R$.
Assume that triangle ABC is acute. Let K, L, M be the midpoints of the sides AB, BC, AC respectively. The point O is the orthocentre of the triangle KLM, which is acute as well as the similar triangle ABC. Thus O lies inside the triangle KLM. Let line MO intersect the segment KL at the point P. We have $AB + BC + AC = 2(AK + KL + LC) = 2(AK + KP) + 2(PL + LC) > 2AP + 2PC > 2AO + 2CO = 4R$ (the last inequality uses that the angles $\angle AOP$ and $\angle COP$ are obtuse). *Lemma 1 is proved.*

Lemma 2. Assume that a polygon is inscribed in a circle of radius R, and the center of the circle lies inside the polygon. Then the perimeter P of the polygon is greater than $4R$.
Proof. Let $A_1 A_2 \dots A_n$ be our polygon. The diagonals $A_1 A_3, A_1 A_4, \dots, A_1 A_{n-1}$ partition it into $n - 2$ triangles. The point O belongs to the interior or the boundary of $A_1 A_i A_{i+1}$. Now Lemma 2 follows from the Lemma 1:

$$P = (A_1 A_2 + \dots + A_{i-1} A_i) + A_i A_{i+1} + (A_{i+1} A_{i+2} + \dots + A_n A_1) \ge$$
$$\ge A_1 A_i + A_i A_{i+1} + A_{i+1} A_1 > 4R\,.$$

Lemma 2 is proved.

Let us return to the problem. Let R denote the circumradius of the circle $A_1A_2\dots A_n$, let R_i denote the circumradius of $B_iA_{i+1}B_{i+1}$ (further we suppose $A_{n+1} \equiv A_1, A_{n+2} \equiv A_2, B_{n+1} \equiv B_1$). The sine law yields $\frac{B_iB_{i+1}}{\sin\angle A_{i+1}} = 2R_i$, $\frac{A_iA_{i+2}}{\sin\angle A_{i+1}} = 2R$, thus $\frac{B_iB_{i+1}}{A_iA_{i+2}} = \frac{2R_i\sin\angle A_{i+1}}{2R\sin\angle A_{i+1}} = \frac{R_i}{R}$.

$$\frac{B_1B_2}{A_1A_3} + \frac{B_2B_3}{A_2A_4} + \dots + \frac{B_nB_1}{A_nA_2} > 1$$
$$\Updownarrow$$
$$\frac{R_1}{R} + \frac{R_2}{R} + \dots + \frac{R_n}{R} > 1$$
$$\Updownarrow$$
$$R_1 + R_2 + \dots + R_n > R\,.$$

In the triangle $B_iA_{i+1}B_{i+1}$ no side can be greater than the diameter of the circumcircle, therefore $B_iA_{i+1} + A_{i+1}B_{i+1} \le 2R_i + 2R_i = 4R_i$ and $R_i \ge (B_iA_{i+1} + A_{i+1}B_{i+1})/4$. Hence it suffices to prove that

$$R < \frac{B_1A_2 + A_2B_2}{4} + \frac{B_2A_3 + A_3B_3}{4} + \dots + \frac{B_nA_1 + A_1B_1}{4} = \frac{P}{4},$$

but this follows from Lemma 2. $\square$