On the Desargues' Involution Theorem

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As the title suggests, this article will deal with one powerful theorem in projective geometry, Desargues' Involution Theorem and its variants. As well as presenting some Olympiad problems which can be solved with this theorem. Readers are expected to be familiar with projective geometry, inversion and some basics of conics.

1 What is involution?

As you may have guessed, this theorem will be deal with involution. In general, involution is any function $f : A \to A$ satisfying $f(f(x)) = x$ for every $x \in A$. But let we restrict a bit more by adding the following conditions.

Definition 1.1. Let $\mathcal{P}$ be the set of all points on a line or a conic. Then the function $f : \mathcal{P} \to \mathcal{P}$ is called involution if and only if it satisfies two conditions.

(i) f preserves cross-ratio. Or for any points $A, B, C, D \in \mathcal{P}$,

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)).$$

(ii) $f(f(A)) = A$ for every point $A \in \mathcal{P}$.

Furthermore, we call a pair $(A, f(A))$ reciprocal pair.

We give some preliminary observations.

Theorem 1.2. *Let $\mathcal{P}$ be the set of all points on a line or a conic. If the function $f : \mathcal{P} \to \mathcal{P}$ preserves cross ratio and have two points $A, A' \in \mathcal{P}$ satisfying $f(A) = A'$ and $f(A') = A$ then f is an involution.*

Proof. Let $P \in \mathcal{P}, Q = f(P)$ and $P' = f(Q)$. Then we have

$$(A, A'; P, Q) = (f(A), f(A'); f(P), f(Q)) = (A', A; Q, P') = (A, A'; P', Q)$$

which implies $P' = P$ as desired. □

Now we will classify involution on a line and involution on a conic. The classification of both type of involutions is very different so we will split the discussion.

1.1 Involution on a line

Some readers may want to see the example of involutions on a line ℓ. The most obvious ones are identity function (trivial) and reflection across a fixed point on ℓ. Thinking a bit more carefully, isogonal-conjugating is an involution as well (it preserves cross-ratio because it's a composition on reflection and projection on to ℓ). But in fact, even inversion is an involution.

Theorem 1.3. *Let ℓ be a line. Then inversion around a fixed point on ℓ is an involution on ℓ.*

Proof. Let A, B, C, D be points on line ℓ and A', B', C', D' be their inverted image. It's suffice to show that

$$(A, B; C, D) = (A', B'; C', D')$$

Let O, p be the center and the power of this inversion. Using directed length, we find that

$$\begin{aligned}
(A', B' : C', D') &= \frac{(\overline{OA'} - \overline{OC'})(\overline{OB'} - \overline{OD'})}{(\overline{OB'} - \overline{OC'})(\overline{OA'} - \overline{OD'})} \\
&= \frac{(\frac{p}{\overline{OA}} - \frac{p}{\overline{OC}})(\frac{p}{\overline{OB}} - \frac{p}{\overline{OD}})}{(\frac{p}{\overline{OB}} - \frac{p}{\overline{OC}})(\frac{p}{\overline{OA}} - \frac{p}{\overline{OD}})} \\
&= \frac{(\overline{OA} - \overline{OC})(\overline{OB} - \overline{OD})}{(\overline{OB} - \overline{OC'})(\overline{OA} - \overline{OD})} \\
&= (A, B; C, D)
\end{aligned}$$

□

Furthermore, the converse of Theorem 2 holds true as well.

Theorem 1.4. *Any involution on a line ℓ is an inversion of some nonzero (possibly negative) power.*

Proof. It is straightforward to verify this theorem algebraically. But we present a synthetic proof by user TinaSprout in Art of Problem Solving.

Let the involution swapping P and a point at infinity on ℓ, points X_1 and X_2, points Y_1 and Y_2. Then we have

$$(P, \infty; X_1, Y_1) = (P, \infty; X_2, Y_2) \implies \frac{PX_1}{PY_1} = \frac{PX_2}{PY_2}$$

Hence $\overline{PX_1} \cdot \overline{PX_2} = \overline{PY_1} \cdot \overline{PY_2}$ (lengths are directed), therefore this involution must be an inversion centered at P. □

1.2 Involution on a conic

Surprisingly, classification involution on a conic is very simple as the following theorem stated.

Theorem 1.5. *Let $\mathcal{C}$ be a conic. Then for any involution f on $\mathcal{C}$, there exists a fixed point P such that f takes point A to the second intersection of PA and $\mathcal{C}$.*

Proof. Since the statement is purely projective, we can take any projective transformation which sends $\mathcal{C}$ to a circle. Therefore without loss of generality, let $\mathcal{C}$ be a circle.

Let $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ be reciprocal pairs of the involution f. Invert around any point $P \in \mathcal{C}$ take $\mathcal{C}$ to a line ℓ. And $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ become reciprocal pairs of an involution f' on a line ℓ. We must show that circles $\odot(PA_1A_2), \odot(PB_1B_2), \odot(PC_1C_2)$ are coaxial.

By Theorem 3, there exists point $K \in \ell$ such that $KA_1 \cdot KA_2 = KB_1 \cdot KB_2 = KC_1 \cdot KC_2$ (lengths are directed). Therefore K lies on radical axis of three circles. But we already know that point $P \neq K$ lies on radical axis of three circles. Hence we are done. □

We also would like to note that we can also project involution from line to conic. This can be a possible way to prove concurrent chords.

2 Desargues' Involution Theorem

Now it may seems that involution is useless because it's just an inversion. But Desargues have shown that any five lines forms three reciprocal pairs of involution. Desargues' Involution Theorem states that.

2.1 The main theorem

Theorem 2.1 (Desargues' Involution Theorem). *Let $ABCD$ be a quadrilateral. A line ℓ intersect lines AB, CD, AD, BC, AC, BD at points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$. A conic $\mathcal{C}$ pass through points A, B, C, D intersects line ℓ at points W_1, W_2. Then pairs $(W_1, W_2), (X_1, X_2), (Y_1, Y_2), (Z_1, Z_2)$ are reciprocal pairs of some involution on ℓ.*

The proof of this theorem requires cross-ratio on a conic. Make sure the reader knows (or convince yourself) that projecting from a line to a conics from a point at that conic preserves cross-ratio.

Proof. [1] Recall that there exists projective transformation f which fix a line ℓ and send points W_1, W_2, X_1 to W_2, W_1, X_2 respectively. By theorem 1, f is an involution hence (X_1, X_2) is a reciprocal pair too. Now it suffices to prove that $(Y_1, Y_2), (Z_1, Z_2)$ are reciprocal pair too. Note that

$$(X_1, Y_1; W_1, W_2) \stackrel{A}{=} (B, D; W_1, W_2) \stackrel{C}{=} (Y_2, X_2; W_1, W_2) = (X_2, Y_2; W_2, W_1)$$

But we already know that $f(X_1) = X_2, f(W_1) = W_2, f(W_2) = W_1$. Therefore $f(Y_1) = Y_2$ and hence (Y_1, Y_2) is a reciprocal pair of f. Similarly (Z_1, Z_2) is reciprocal pair of f too and we are done. □

The degenerate case of this theorem is true. Here we present the three points and two points version of this theorem. Convince yourself that the following statements are true.

Theorem 2.2 (2 Points Desargues Involution). *Let A, B be points on a conic $\mathcal{C}$. A line ℓ intersects lines AB at X and intersects lines tangent to $\mathcal{C}$ at A, B at Y_1, Y_2. Line ℓ also intersects $\mathcal{C}$ at W_1, W_2. Then pairs (W_1, W_2), (X, X), (Y_1, Y_2) are reciprocal pairs of some involution on ℓ.*

Theorem 2.3 (3 Points Desargues Involution). *Let ABC be a triangle inscribed in a conic $\mathcal{C}$. A line ℓ intersects lines AB, AC, BC at X_1, X_2, Y_1 and intersects lines tangent to $\mathcal{C}$ at A at Y_2. Line ℓ also intersects $\mathcal{C}$ at W_1, W_2. Then pairs (W_1, W_2), (X_1, X_2), (Y_1, Y_2) are reciprocal pairs of some involution on ℓ.*

2.2 Dual of Desargues' Involuion Theorem

You should be familiar with pole-polar concept which turn colinearity into concurrency, tangent into lying on a circle. This makes a purely projective theorem come in pairs (e.g. Pascal's and Brianchon's, Newton's and Brokard's). Dersargues' Involution Theorem also have it's pair (or dual).

Definition 2.4. Let P be a point on the plane. Let $\mathcal{L}$ be the set of all line containing P. Then $f : \mathcal{L} \to \mathcal{L}$ is involution on a pencil of lines if and only if.

(i) For every $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD} \in \mathcal{L}$, we have

$$(\overline{PA}, \overline{PB}; \overline{PC}, \overline{PD}) = (f(\overline{PA}), f(\overline{PB}); f(\overline{PC}), f(\overline{PD}))$$

(ii) $f(f(\ell)) = \ell$ for every $\ell \in \mathcal{L}$. Furthermore, we call a pair $(\ell, f(\ell))$ reciprocal pair.

Of course, if we have an involution on a pencil, we can project onto a line to get an involution on a line.

Theorem 2.5 (Dual of Desargues' Involution Theorem)**.** *Let P, A, B, C, D be points on a plane with $\overline{AB} \cap \overline{CD} = E, \overline{AD} \cap \overline{BC} = F$. Let a conic $\mathcal{C}$ tangent to lines AB, CD, AD, BC. Let $\overline{PX}, \overline{PY}$ are the tangent line from P to $\mathcal{C}$. Then $(\overline{PX}, \overline{PY}), (\overline{PA}, \overline{PC}), (\overline{PB}, \overline{PD}), (\overline{PE}, \overline{PF})$ are reciprocal pairs of some involution on pencil of lines pass through P.*

Proof. Take a pole-polar transformation with respect to $\mathcal{C}$, this is equivalent to Desargues' Involution Theorem. □

Usually, this dual statement is more useful than the original one and it is usually stated as just Desargues' Involution Theorem. The three points and two points versions also true as we are going to state them here.

Theorem 2.6 (Dual of 2 Points Desargues Involution)**.** *Let A, B be points on a conic $\mathcal{C}$ and let P be a point on a plane. Lines tangent to $\mathcal{C}$ at A, B intersect at X. Lines pass through P tangent to $\mathcal{C}$ at Y, Z. Then pairs $(\overline{PY}, \overline{PZ})$, $(\overline{PX}, \overline{PX})$, $(\overline{PA}, \overline{PB})$ are reciprocal pairs of some involution on the pencil of lines pass through P.*

Theorem 2.7 (Dual of 3 Points Desargues Involution)**.** *Let ABC be a triangle and P be a point in a plane. A conic $\mathcal{C}$ which tangent to lines BC, AC, AB tangent to BC at D. Let P be a point on a plane. Let lines through P which tangent to $\mathcal{C}$ tangent at X, Y respectively Then $(\overline{PX}, \overline{PY})$, $(\overline{PA}, \overline{PD})$, $(\overline{PB}, \overline{PC})$ are reciprocal pairs of some involution on pencil of lines pass through P.*

Note : Usually, we only use the special case where $\mathcal{C}$ is a circle.

3 Examples

First, let us nuke a fifth problem on USAMO with this theorem.

Example 3.1 (USAMO 2012 P5)**.** Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A_1, B_1, C_1 be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A_1, B_1, C_1 are collinear.

Proof. By reflection, there is an involution swapping $(\overline{PA}, \overline{PA_1}), (\overline{PB}, \overline{PB_1}), (\overline{PC}, \overline{PC_1})$. Now let $C_1' = \overline{A_1B_1} \cap \overline{AB}$. By Desargues' Involution theorem, this involution must swap $(\overline{PC}, \overline{PC_1'})$. Hence $C_1 = C_1'$ and we are done. □

Let we present the use of involution on a conic. This problem is actually very straightforward with involution but very difficult to solve in other way.

Example 3.2 ([2])**.** Let ABC and DEF be two triangles which share an incircle ω and circumcircle γ. Let L be the tangency point of EF on ω and define K similarly on BC. Select $N \equiv AL \cap \gamma$ and $M \equiv DK \cap \gamma$. Show that lines AM, EF, BC, ND are concurrent.

Proof. By Desargues' Involution theorem, there exists an involution swapping $(\overline{DA}, \overline{DM})$, $(\overline{DB}, \overline{DC})$, $(\overline{DE}, \overline{DF})$. Projecting this involution to γ, we have an involution swapping (A, M), (B, C), (E, F). By theorem 3, lines AM, EF, BC are concurrent. Similarly EF, BC, ND are concurrent and we are done. □

We would like to close this article with the following extremely difficult problem from Taiwan TST. No one has solved this during exam but it has very short and elegant solution with Desargues' Involution Theorem.

Example 3.3 (Taiwan TST3 2014 P3)**.** Let M be any point on the circumcircle of ΔABC. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that the circumcircle of ΔMX_1X_2 intersects the circumcircle of ΔABC again at the tangency point of the A-mixtilinear incircle.

Solution. Let $\odot(I)$ denote the incircle of ΔABC. Let $\odot(I)$ touch BC at D and let T be the tangency point of the A-mixtilinear incircle.

By Desargues' Involution Theorem, there is an involution swapping $(\overline{MX_1}, \overline{MX_2})$, $(\overline{MB}, \overline{MC})$, $(\overline{MD}, \overline{MA})$. Projecting onto the line BC and let $\overline{AM} \cap \overline{BC} = N$, there exists an involution swapping $(D, N), (B, C), (X_1, X_2)$.

By Theorem 3, let K be the center of inversion and consider the circles $\odot(ABMC)$, $\odot(MDN)$ $\odot(MX_1X_2)$. We have $KB \cdot KC = KD \cdot KN = KX_1 \cdot KX_2$ hence MK is the radical axis of three circles. Therefore they are coaxial and must meet at other point.

Now it suffices to show that M, D, N, T are concyclic. To do that, extend $\overline{TD}$ to meet $\odot(ABC)$ at A_1. It's well known (by $\angle BTD = \angle CTA$) that $AA_1 \parallel BC$. Hence we are done because $\angle ANB = \angle A_1AM = \angle A_1TM = \angle DTM$. $\square$

4 Practice Problems

Desargues' Involution theorem can be manipulated for many different situations. Sometimes it trivializes problem but sometimes it makes only a tiny progress. Most of the given problems are difficult geometry problem but it has different difficulty when knowing Desargues' Involution theorem. Have fun.

Problem 4.1 (China TST2 2017 P3)**.** Let $ABCD$ be a quadrilateral and let l be a line. Let l intersect the lines AB, CD, BC, DA, AC, BD at points X, X', Y, Y', Z, Z' respectively. Given that these six points on l are in the order X, Y, Z, X', Y', Z', show that the circles with diameter XX', YY', ZZ' are coaxal.

Problem 4.2 (Serbia MO 2017 P6)**.** Let k be the circumcircle of $\triangle ABC$ and let k_a be A-excircle . Let the two common tangents of k, k_a cut BC in P, Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

Problem 4.3 (IMO shortlist 2005 G6)**.** Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points X and Y. Let the lines through X and Y, parallel to BC, intersect the incircle γ again in two points X_1 and Y_1. Let the lines AX_1 and AY_1 intersect BC again at the points P and Q. Prove that $BP = CQ$.

Problem 4.4 ([3])**.** Let ABC be a triangle with orthocenter H and circumcircle Ω centered at O. Let M_A, M_B, M_C be the midpoints of sides BC, CA, AB. Lines AM_A, BM_B, CM_C intersect Ω again at P_A, P_B, P_C. Rays M_AH, M_BH, M_CH intersect Ω at Q_A, Q_B, Q_C. Prove that lines P_AQ_A, P_BQ_B, P_CQ_C and OH are concurrent.

Problem 4.5 (IMO Shortlist 2012 G8)**.** Let ABC be a triangle with circumcenter O and ℓ a line. Denote by P the foot from O to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circles $\odot(AXP)$, $\odot(BYP)$ and $\odot(CZP)$ are coaxial.

Problem 4.6 (China TST1 2017 P5)**.** In the non-isosceles triangle ABC, M_A, M_B, M_C is the midpoint of side BC, CA, AB. The line (different from line BC) through M_A that is tangent to the incircle of triangle ΔABC intersect line M_BM_C at X. Define Y, Z similarly. Prove that points X, Y, Z are collinear.

Problem 4.7 (IMO Shortlist 2015 G7)**.** Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA. Let the segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Problem 4.8 (IMO 2008 P6)**.** Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to rays BA, BC, AD, CD. Prove that the exsimilicenter of ω_1 and ω_2 lies on ω.

References

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